

Squeezed coherent state undergoing a continuous nondemolition observation

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Abstract

The time evolution of a squeezed coherent state conditioned by the results of a single and double heterodyne measurement is discussed. The mean values of quadratures as well as the dynamics of quadrature uncertainties have been obtained within the framework of the theory of continuous measurements based on filtration equations. It has been found that while the mean values depend on the measured noise, the uncertainties in the optical quadratures are deterministic. Explicit solutions for the latter have been provided. Finally, a time development of the squeeze parameter for the posterior squeezed coherent state has been found.

Keywords: Nondemolition quantum measurements, Quantum filtration, Heterodyne measurement, Squeezed coherent state

1. Introduction

The problem of continuous measurements in quantum systems is one of the most challenging fundamental issues of modern theoretical physics [1, 2]. The quantum filtering theory developed by Belavkin [3–5] makes it possible to describe the dynamics of a quantum system continuously observed in time. The rôle of a measuring apparatus is played here by a Bose field which in quantum optics can be treated as an approximation to the electromagnetic field. The interaction between the quantum system in question (system S) and the reservoir (the Bose field) is taken in the Markovian approximation, i.e. the correlation time of the reservoir is much shorter than the time scale of the dynamics of S . The continuous trajectory of results of the observation of the output Bose field (Bose field after interaction with S) determines the conditioned evolution of S (the posterior state). The main motivation for determining the conditional state of a quantum system is developing methods of the quantum feedback control. The quantum trajectory has been essential to the design of a quantum control algorithm [6].

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In this article, we present the posterior evolution of an optical cavity mode coupled to the outside radiation mode by one or two partially transmitting mirrors. The time-development of a state of an indirectly observed system is conditioned by a trajectory of the results of a single and double heterodyne measurement. We assume the coupling system's operator proportional to the annihilation operator. In the paper we discuss the analytical solutions to the filtering equation for the system being initially in a squeezed coherent state. In order to prove that such a state is preserved under the considered continuous diffusion observation we use the linear version of the quantum filtering equation derived in [4].

The exact solutions to the filtering equation for the initial Gaussian states and a diffusion observation were given, for instance, in [5, 7–10]. The solution to the quantum filtering equation for a harmonic oscillator in an arbitrary initial state undergoing the heterodyne observation was discussed by Carmichael in [11].

The main part of our work is organized as follows. In Section 2 the mathematical model is described. Section 3 contains the analysis of the double heterodyne detection problem. Section 4 is devoted to the single heterodyne detection. Some final remarks are given in Section 5.

2. Model

We consider a single cavity mode of the electromagnetic field (a system \mathcal{S}) interacting with two independent components $B_n(t)$, ($n = 1, 2$) of the Bose field being initially in the vacuum state. The unitary operator, $U(t)$, describing the evolution of the whole system (the system \mathcal{S} plus Bose field) satisfies the Ito quantum stochastic differential equation (QSDE):

$$\begin{aligned} dU(t) = & \left[-\left(\frac{i}{\hbar} H + \sum_{n=1}^2 \frac{\mu_n}{4} a^\dagger a \right) dt \right. \\ & \left. + \sum_{n=1}^2 \sqrt{\frac{\mu_n}{2}} \left(a dB_n^\dagger(t) - a^\dagger dB_n(t) \right) \right] U(t), \\ U(0) = & I, \end{aligned} \tag{1}$$

where $H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$ is the hamiltonian of \mathcal{S} , a stands for the annihilation operator of \mathcal{S} and $\mu_n > 0$ ($n = 1, 2$) are coupling constants. Eq. (1) is written in the interaction picture with respect to the free dynamics of the Bose field. The discussion of the physical assumptions leading to this evolution one can find, for instance, in [12, 13]. In brief, the interaction hamiltonian is taken linear in the field operators, the rotating-wave approximation (RWA) is made and a flat and broad spectrum of the reservoir is assumed.

The Bose field provides a possibility of a continuous indirect observation of \mathcal{S} . We suppose that the information about the systems is gained by using a double heterodyne detection scheme, thus we consider a simultaneous measurement of the two output

processes

$$\begin{aligned} Q_n^{\text{out}}(t) = & \int_0^t \left(e^{i\phi_n(t')} dB_n^\dagger(t') + e^{-i\phi_n(t')} dB_n(t') \right) \\ & + \sqrt{\mu_n/2} \operatorname{Re} \left(e^{-i\phi_n(t')} a \right) dt', \quad n = 1, 2, \end{aligned} \quad (2)$$

where $\phi(t) = \phi_0 + \vartheta t$ [14, 15]. These output processes satisfy the Belavkin's nondemolition condition, namely

$$[Q_n^{\text{out}}(s), U^\dagger(t) Z U(t)] = 0 \quad \forall s \leq t, \quad (3)$$

where Z is any operator of the system \mathcal{S} . Furthermore, the output processes (2), due to their Hermiticity and self-commutativity,

$$[Q_n^{\text{out}}(t), Q_n^{\text{out}}(t')] = 0 \quad \forall t, t' \geq 0, \quad (4)$$

can be treated as the classical Wiener processes.

The time development of the posterior unnormalized wave function $\widehat{\psi}(t)$ of \mathcal{S} , corresponding to the trajectory of the observed processes (2) up to t , is given by the Belavkin linear filtering equation of the form

$$\begin{aligned} d\widehat{\psi}(t) = & - \left(\frac{i}{\hbar} H + \sum_{n=1}^2 \frac{\mu_n}{4} a^\dagger a \right) \widehat{\psi}(t) dt \\ & + \sum_{n=1}^2 \sqrt{\frac{\mu_n}{2}} a e^{-i\phi_n(t)} \widehat{\psi}(t) dQ_n(t), \quad \widehat{\psi}(0) = \psi, \end{aligned} \quad (5)$$

where $Q_n(t)$ ($n = 1, 2$) are Wiener processes for which $dQ_n(t)dQ_m(t) = \delta_{nm}dt$. Let us recall that the posterior mean value of any operator Z of the system \mathcal{S} reads in terms of unnormalized wave function $\widehat{\psi}(t)$ satisfying the linear filtering equation

$$\langle Z \rangle_t = \frac{\langle \widehat{\psi}(t) | Z \widehat{\psi}(t) \rangle}{\langle \widehat{\psi}(t) | \widehat{\psi}(t) \rangle}. \quad (6)$$

For more details on the rôle of the linear filtering equation for the diffusion observation in quantum mechanics, its derivation, and some exemplary solutions see for instance [4, 16].

3. Double heterodyne detection

Let us put: $\mu_n = \mu$ ($n = 1, 2$), $\phi_2(t) - \phi_1(t) = \pi/2$, and $\phi_1(t) = \phi(t)$. With these assumptions, the linear filtering equation for the double heterodyne measurement takes the form

$$\begin{aligned} d\widehat{\psi}(t) = & - \left(\frac{i}{\hbar} H + \frac{\mu}{2} a^\dagger a \right) \widehat{\psi}(t) dt + \sqrt{\mu} a e^{-i\phi(t)} \widehat{\psi}(t) dQ(t), \\ \widehat{\psi}(0) = & \psi, \end{aligned} \quad (7)$$

where $dQ(t) = \frac{1}{\sqrt{2}} (dQ_1(t) - i dQ_2(t))$ is the complex diffusion process such that $(dQ(t))^2 = 0$ and $|dQ^\dagger(t)|^2 = dt$.

Let us discuss the time development of the posterior wave function, assuming that the initial state of S is a squeezed coherent state [19],

$$\widehat{\psi}(0) = S(\xi_0)D(\alpha_0)|0\rangle = S(\xi_0)|\alpha_0\rangle = |\xi_0, \alpha_0\rangle, \quad (8)$$

where

$$D(\alpha_0) = \exp(\alpha_0 a^\dagger - \bar{\alpha}_0 a), \quad (9)$$

and

$$S(\xi_0) = \exp\left(\frac{1}{2}\bar{\xi}_0 a^2 - \frac{1}{2}\xi_0 (a^\dagger)^2\right), \quad \xi_0 = e^{i\theta_0} \varrho_0 \in \mathbb{C}. \quad (10)$$

The amount of squeeze is described by the modulus ρ_0 of the squeeze parameter ξ_0 , whereas the phase θ_0 specifies the angle of the squeeze axis in the phase space. Obviously, for $\xi_0 = 0$ the state (8) becomes the coherent state $|\alpha_0\rangle$, and for $\alpha_0 = 0$ and $\xi_0 \neq 0$ it takes the form of the squeezed vacuum state, $S(\xi_0)|0\rangle$. Let us recall that the expectation values of the quadratures vanish while the expectation value of the photon number operator is nonzero in the state $S(\xi_0)|0\rangle$.

We shall prove that the solution to Eg. (7) corresponding to the initial state (8) can be written as

$$\widehat{\psi}(t) = l(t)S(\xi(t))|\alpha(t)\rangle. \quad (11)$$

The proof is rather standard but still quite cumbersome, therefore we shall present its outline augmented with some computational details and partial results.

Substituting the postulated solution (11) into Eg. (7), writing both sides of the equation in terms of linearly independent vectors $\{|\alpha\rangle, \frac{\partial|\alpha\rangle}{\partial\alpha}, \frac{\partial^2|\alpha\rangle}{\partial\alpha^2}\}$ and comparing the coefficients of the corresponding vectors yield the consistent system of differential equations for the functions: $\varrho(t)$, $\theta(t)$, $\alpha(t)$, $l(t)$. The solution to this system corresponding to the initial state (8) uniquely defines the solution to Eg. (7) in the form (11). Let us pay attention to several steps of these calculations.

To calculate the increment $d\widehat{\psi}(t)$ one has to calculate the increment $dS(\xi(t)) = S(\xi(t+dt)) - S(\xi(t))$. To this end it is convenient to make use of the normally ordered form [20] of the squeeze operator, $S(\xi)$,

$$\begin{aligned} S(\xi) &= (\cosh \varrho)^{-1/2} \exp\left[-\Gamma (a^\dagger)^2/2\right] \\ &\times \exp\left[-\ln(\cosh \varrho) a^\dagger a\right] \exp\left[\bar{\Gamma} a^2/2\right], \end{aligned} \quad (12)$$

where

$$\Gamma = e^{i\theta} \tanh \varrho. \quad (13)$$

Left-multiplying Eq. (7) by $S^\dagger(\xi(t))$ and using the unitary transformation

$$\begin{aligned} S^\dagger(\xi) (a^\dagger)^2 S(\xi) &= (a^\dagger)^2 \cosh^2 \varrho \\ &- (2a^\dagger a + 1) e^{-i\theta} \sinh \varrho \cosh \varrho + a^2 e^{-2i\theta} \sinh^2 \varrho, \end{aligned} \quad (14)$$

one gets

$$\begin{aligned}
S^\dagger(\xi(t))dS(\xi(t)) &= \frac{1}{2} \tanh \varrho(t) d\varrho(t) \left[2\bar{\Gamma}(t)a^2 - 2a^\dagger a - 1 \right] \\
&+ \frac{d\bar{\Gamma}(t)}{2} a^2 - \frac{d\Gamma(t)}{2} \left[(a^\dagger)^2 \cosh^2 \varrho(t) \right. \\
&\left. - (2a^\dagger a + 1) e^{-i\theta(t)} \sinh \varrho(t) \cosh \varrho(t) + a^2 e^{-2i\theta(t)} \sinh^2 \varrho(t) \right].
\end{aligned} \tag{15}$$

By the virtue of applying the formulae

$$S^\dagger(\xi)aS(\xi) = a \cosh \varrho - a^\dagger e^{i\theta} \sinh \varrho, \tag{16}$$

$$\begin{aligned}
S^\dagger(\xi)a^\dagger aS(\xi) &= a^\dagger a \cosh^2 \varrho + (a^\dagger a + 1) \sinh^2 \varrho \\
&- \left[a^2 e^{-i\theta} + (a^\dagger)^2 e^{i\theta} \right] \sinh \varrho \cosh \varrho
\end{aligned} \tag{17}$$

and

$$a^\dagger |\alpha\rangle = \frac{\partial |\alpha\rangle}{\partial \alpha} + \frac{1}{2} \bar{\alpha} |\alpha\rangle, \tag{18}$$

$$(a^\dagger)^2 |\alpha\rangle = \frac{\partial^2 |\alpha\rangle}{\partial \alpha^2} + \bar{\alpha} \frac{\partial |\alpha\rangle}{\partial \alpha} + \frac{1}{4} \bar{\alpha}^2 |\alpha\rangle, \tag{19}$$

one is able to rewrite the equation in terms of the vectors $\{|\alpha\rangle, \frac{\partial |\alpha\rangle}{\partial \alpha}, \frac{\partial^2 |\alpha\rangle}{\partial \alpha^2}\}$. As these vectors are linearly independent, the comparison of the expansion coefficients yields the set of differential equations for the functions θ , ϱ , α , and l :

$$\begin{aligned}
d\theta(t) &= -2\omega dt, \quad d\varrho(t) = -\mu \sinh \varrho(t) \cosh \varrho(t) dt, \\
d\alpha(t) &= \left[-\left(i\omega + \frac{\mu}{2} \right) - \mu \sinh^2 \varrho(t) \right] \alpha(t) dt \\
&- \sqrt{\mu} e^{i\theta(t)} \sinh \varrho(t) e^{-i\phi(t)} dQ(t), \\
\frac{dl(t)}{l(t)} &= -\frac{i\omega}{2} dt + \frac{1}{2} d|\alpha(t)|^2 + \frac{\mu}{4} \sinh^2 \varrho(t) |\alpha(t)|^2 dt \\
&+ \sqrt{\mu} \alpha(t) \cosh \varrho(t) e^{-i\phi(t)} dQ(t) - \frac{\mu}{2} \sinh^2 \varrho(t) dt \\
&+ \frac{\mu}{2} \alpha^2(t) e^{-i\theta(t)} \sinh \varrho(t) \cosh \varrho(t) dt,
\end{aligned} \tag{20}$$

with the initial condition $l(0) = 1$, $\alpha(0) = \alpha_0$, $\theta(0) = \theta_0$, $\varrho(0) = \varrho_0$.

The solution to the system (20) reads

$$\begin{aligned}
\theta(t) &= \theta_0 - 2\omega t, \quad \varrho(t) = \operatorname{ar tanh} \left(e^{-\mu t} \tanh \varrho_0 \right), \\
\alpha(t) &= e^{-(i\omega + \frac{\mu}{2})t} \frac{\cosh \varrho(t)}{\cosh \varrho_0} \left(\alpha_0 \right. \\
&\left. - \sqrt{\mu} e^{i\theta_0} \sinh \varrho_0 \int_0^t e^{-(\omega + \frac{\mu}{2})t'} e^{-i\phi(t')} dQ(t') \right), \\
l(t) &= \sqrt{\frac{\cosh \varrho(t)}{\cosh \varrho_0}} \exp \left[-\frac{i\omega t}{2} + \frac{1}{2} (|\alpha(t)|^2 - |\alpha_0|^2) + \chi(t) \right],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}\chi(t) = & \sqrt{\mu} \int_0^t \left(\alpha(t') \cosh \varrho(t') e^{-i\phi(t')} dQ(t') \right. \\ & \left. + \sqrt{\mu} e^{-i\theta(t')} \alpha^2(t') \sinh \varrho(t') \cosh \varrho(t') dt' \right) .\end{aligned}$$

Therefore the posterior mean values of optical quadratures $X = (a + a^\dagger)/2$ and $Y = (a - a^\dagger)/2i$ for the posterior wave function of the form (11), given by

$$\langle X \rangle_t = \text{Re} \left(\alpha(t) \cosh \varrho(t) - \bar{\alpha}(t) e^{i\theta(t)} \sinh \varrho(t) \right) , \quad (22)$$

$$\langle Y \rangle_t = \text{Im} \left(\alpha(t) \cosh \varrho(t) + \alpha(t) e^{-i\theta(t)} \sinh \varrho(t) \right) , \quad (23)$$

depend on the measured noise, whereas the uncertainties in the optical quadratures $\Delta X(t)$ and $\Delta Y(t)$ are deterministic. One has:

$$\Delta X(t) = \frac{1}{2} \left[1 + C(t) \left(e^{-\mu t} \tanh \varrho_0 - \cos(\theta_0 - 2\omega t) \right) \right]^{1/2} , \quad (24)$$

$$\Delta Y(t) = \frac{1}{2} \left[1 + C(t) \left(e^{-\mu t} \tanh \varrho_0 + \cos(\theta_0 - 2\omega t) \right) \right]^{1/2} , \quad (25)$$

where

$$C(t) = \frac{2e^{-\mu t} \tanh \varrho_0}{1 - e^{-2\mu t} \tanh^2 \varrho_0} .$$

The squeezing of X occurs for

$$\cos(\theta_0 - 2\omega t) > e^{-\mu t} \tanh \varrho_0 , \quad (26)$$

while the squeezing of Y occurs for

$$\cos(\theta_0 - 2\omega t) < -e^{-\mu t} \tanh \varrho_0 . \quad (27)$$

The time dependence of the uncertainties ΔX and ΔY has been illustrated by the parametric plots presented in Fig. 1. They show the dynamics of ΔX and ΔY as functions of the dimensionless time $\tau = \omega t$ ($0 \leq \tau \leq 100$) for $\mu = 0.01\omega$, $\theta_0 = 0$, and three values of ρ_0 . The pictures for different values of ρ_0 differ mostly for short times, otherwise they are qualitatively very similar. From the above formulae as well as from the figure it is clear that the dynamics of quadrature uncertainties is such that the system switches back and forth from being squeezed in one of the quadratures but not in the other while passing also through the region where there is no squeezing at all. Asymptotically, the system approaches the vacuum state (with $\Delta X = \Delta Y = \frac{1}{2}$). An interesting feature of the envelopes of the displayed curves is that the region where they are concave shrinks as time grows, and the envelopes finally become convex.

A similar result concerning the uncertainties in the posterior optical quadratures as well as the squeezing coefficients was obtained in [16] where subharmonic generation from the vacuum was studied: the uncertainties and $\eta(t)$ were found to be deterministic, while $\alpha(t)$ was generated from the output noise.

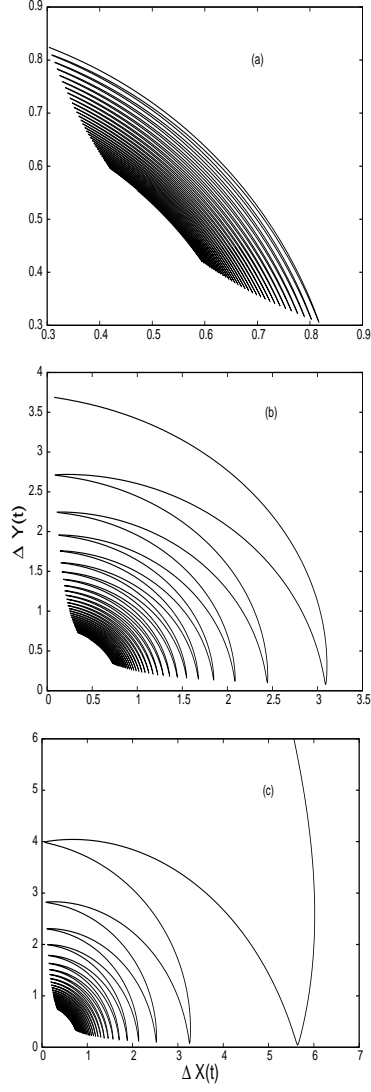


Figure 1: Time dependence of the uncertainties ΔX and ΔY as given by Eqs. (24-25). The dependence of ΔX and ΔY on the dimensionless time $\tau = \omega t$ is displayed for $\mu = 0.01\omega$, $\theta_0 = 0$, $\vartheta = 0.05$ and for three values of ρ_0 : 0.5 (a), 2.0 (b), and 8.0 (c).

It may be appropriate at this point to recall that the posterior wave function satisfying the linear version of the filtering equation is normalized to the probability density of the output (observed) diffusion process with respect to the standard Wiener measure of the input Wiener diffusion process. In the considered case of the double heterodyne observation Eq. (11) implies $\|\widehat{\psi}(t)\|^2 = |l(t)|^2$, therefore $|l(t)|^2$ is the probability density of the complex Wiener process $\widehat{Q}(t)$ with respect to the standard Wiener measure of the complex input process $Q(t)$. Note that the function $l(t)$ is essential in the considered continuous observation. For example, as one can see from (21) that $l(t)$ depends on all the parameters of an initial squeezed coherent state. If the values of some of them or even all the values of these parameters are unknown, they can be determined from $l(t)$.

We have proved that the filtering equation makes it possible to study the time-development of a squeezed coherent state. This would not be possible with the help of the master equation which can be obtained from the filtering equation for the mixed posterior state obtained from the Eq. (7) by taking the stochastic average over all possible trajectories of the observed process. Though the master equation (of the same form for both cases of the heterodyne observation considered in the paper) preserves an initially coherent state [17], it does not preserve the squeezed coherent one [18].

4. Single balance heterodyne detection

The filtering equation for a single balance heterodyne detection

$$\begin{aligned} d\widehat{\psi}(t) &= -\left(\frac{i}{\hbar}H + \frac{\mu}{2}a^\dagger a\right)\widehat{\psi}(t)dt + \sqrt{\mu}a e^{-i\phi(t)}\widehat{\psi}(t)dQ(t), \\ \widehat{\psi}(0) &= \psi, \end{aligned} \quad (28)$$

can be obtained from Eq. (5) by putting $\mu_1 = 2\mu$, $\mu_2 = 0$, $\phi_1(t) = \phi(t)$, $Q_1(t) = Q(t)$.

We shall prove that the squeezed coherent state is preserved under the diffusion observation. For this purpose we employ the property

$$S(\xi)aS^\dagger(\xi)|\xi, \alpha\rangle = \alpha|\xi, \alpha\rangle. \quad (29)$$

Making use of the Baker-Hausdorff formula one can get the unitary transform of the operator a [20]

$$S(\xi)aS^\dagger(\xi) = a\Gamma_1 + a^\dagger\Gamma_2, \quad (30)$$

where $\Gamma_1 = \cosh \varrho$, $\Gamma_2 = e^{i\theta} \sinh \varrho$.

Let us notice that if the system remains in the squeezed coherent state (11) at any time instant $t \geq 0$, then the following relations have to be satisfied

$$S(\xi(t))aS^\dagger(\xi(t))\widehat{\psi}(t) = [a\Gamma_1(t) + a^\dagger\Gamma_2(t)]\widehat{\psi}(t), \quad (31)$$

$$\begin{aligned} S(\xi(t+dt))aS^\dagger(\xi(t+dt))\widehat{\psi}(t+dt) \\ = [a\Gamma_1(t+dt) + a^\dagger\Gamma_2(t+dt)]\widehat{\psi}(t+dt). \end{aligned} \quad (32)$$

Eqs. (31) and (32) can be reduced to the single condition

$$\begin{aligned} & \left[a(\Gamma_1(t) + d\Gamma_1(t)) + a^\dagger(\Gamma_2(t) + d\Gamma_2(t)) - \alpha(t) - d\alpha(t) \right] d\widehat{\psi}(t) \\ & + \left(a d\Gamma_1(t) + a^\dagger d\Gamma_2(t) - d\alpha(t) \right) \widehat{\psi}(t) = 0. \end{aligned} \quad (33)$$

Then by insertion of $d\widehat{\psi}(t)$ from Eq. (28) into Eq. (33) we obtain the set of the differential equations

$$\begin{aligned} & \alpha(t)\Gamma_1(t) \left[-\Gamma_1(t) \left(i\omega + \frac{\mu}{2} \right) dt + d\Gamma_1(t) + \mu e^{-2i\phi(t)} \Gamma_2(t) dt \right] \\ & - d\alpha(t) - \alpha(t) \bar{\Gamma}_2(t) \left[\Gamma_2(t) \left(i\omega + \frac{\mu}{2} \right) dt + d\Gamma_2(t) \right] \\ & - \sqrt{\mu} \Gamma_2(t) e^{-i\phi(t)} dQ(t) = 0, \end{aligned} \quad (34)$$

$$\begin{aligned} & \Gamma_2(t) \left[-\Gamma_1(t) \left(i\omega + \frac{\mu}{2} \right) dt + d\Gamma_1(t) + \mu e^{-2i\phi(t)} \Gamma_2(t) dt \right] \\ & - \Gamma_1(t) \left[\Gamma_2(t) \left(i\omega + \frac{\mu}{2} \right) dt + d\Gamma_2(t) \right] = 0 \end{aligned} \quad (35)$$

with the initial condition: $\Gamma_1(0) = \cosh \varrho_0$, $\Gamma_2(0) = e^{i\theta_0} \sinh \varrho_0$, $\alpha(0) = \alpha_0$, and this completes the proof.

The equation (35) imply that the function $\Gamma(t) = e^{i\theta(t)} \tanh \varrho(t)$ satisfies the Riccati differential equation of the form

$$\begin{aligned} \frac{d}{dt} \Gamma(t) &= -2 \left(i\omega + \frac{\mu}{2} \right) \Gamma(t) + \mu e^{-2i\phi(t)} \Gamma^2(t), \\ \Gamma(0) &= e^{i\theta_0} \tanh \varrho_0. \end{aligned} \quad (36)$$

Hence the posterior uncertainties of quadratures for the posterior squeezed coherent state, given by the formulae

$$\Delta X(t) = (4\text{Re}\kappa(t))^{-1/2}, \quad (37)$$

$$\Delta Y(t) = |\kappa(t)| (4\text{Re}\kappa(t))^{-1/2}, \quad (38)$$

where

$$\kappa(t) = \frac{1 + \Gamma(t)}{1 - \Gamma(t)} \quad (39)$$

do not depend on the measured noise, as before. The general solution to Eq. (36) can be written as

$$\Gamma(t) = \frac{\Gamma(0)e^{-(2i\omega + \mu)t}}{1 - \mu \Gamma(0) \int_0^t e^{-(2i\omega + \mu)t'} e^{-2i\phi(t')} dt'}. \quad (40)$$

In particular, for the phase $\phi(t) = \pi/2 + \vartheta t$, we obtain

$$\Gamma(t) = \frac{(2i\omega + 2i\vartheta + \mu)\Gamma(0)}{e^{(2i\omega + \mu)t} [2i\omega + 2i\vartheta + \mu(1 + \Gamma(0))] - \mu \Gamma(0) e^{-2i\vartheta t}}. \quad (41)$$

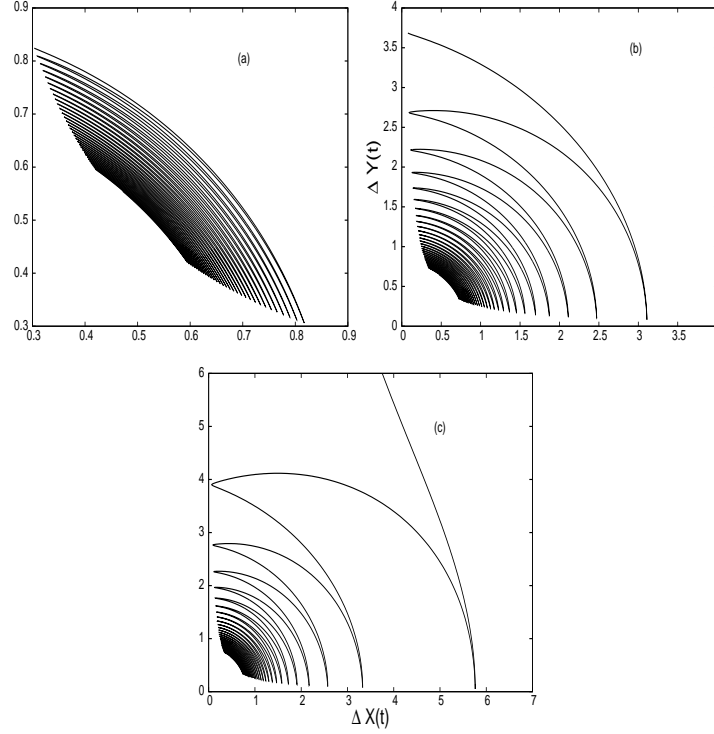


Figure 2: Same as Fig. 1 except for Eqs. (36-37) with $\phi(t) = \pi/2 + \vartheta t$.

The time dependence of the uncertainties ΔX and ΔY computed above has been illustrated in Fig. 2 showing the dynamics of ΔX and ΔY (with the same values of all parameters as in Section 3). It is very hard to find any qualitative difference between the corresponding parts of Fig. 2 and Fig. 1. Such a difference exists only for small times. This is because the crucial variable $\Gamma(t)$ becomes asymptotically exponential for large times, that is, it behaves in the same way as in the case considered in Section 3.

To derive the differential equation for the coefficient $l(t)$ one has to insert the state $l(t)|\xi(t), \alpha(t)\rangle$ to the filtering equation (28). The calculations are more complicated in comparison with these of Section 3. Finally one gets:

$$\begin{aligned}
 l(t) = \exp & \left[-\frac{i\omega t}{2} + \frac{1}{2}(|\alpha(t)|^2 - |\alpha_0|^2) \right. \\
 & -\mu \int_0^t \left(\frac{|\Gamma(t)|^2}{2(1-|\Gamma(t)|^2)} - \frac{\overline{\Gamma(t)}\alpha^2(t)}{1-|\Gamma(t)|^2} \right) dt \\
 & \left. -\frac{\mu}{2} \int_0^t \frac{e^{-2i\phi(t)}}{1-|\Gamma(t)|^2} \left(\frac{1}{2} |\Gamma(t)|^2 \Gamma(t) + \alpha^2(t) \right) dt \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\mu}{2} \int_0^t \frac{(\overline{\Gamma(t)})^2 e^{2i\phi(t)}}{1 - |\Gamma(t)|^2} \left(\frac{1}{2} \Gamma(t) - \alpha^2(t) \right) dt \\
& + \sqrt{\mu} \int_0^t \frac{e^{-i\phi(t)} \alpha(t)}{\sqrt{1 - |\Gamma(t)|^2}} dQ(t) \Big]
\end{aligned} \tag{42}$$

with

$$\begin{aligned}
\alpha(t) = & \frac{1}{\sqrt{1 - |\Gamma(t)|^2}} \left[\alpha_0 \sqrt{1 - |\Gamma(0)|^2} \right. \\
& \times \exp \left(-i\omega t - \frac{\mu}{2} t + \mu \int_0^t e^{-2i\phi(t')} \Gamma(t') dt' \right) \\
& - \sqrt{\mu} \int_0^t \exp \left(- \left(i\omega + \frac{\mu}{2} \right) (t - s) \right. \\
& \left. \left. + \mu \int_s^t e^{-2i\phi(t')} \Gamma(t') dt' \right) e^{-i\phi(s)} \Gamma(s) dQ(s) \right].
\end{aligned} \tag{43}$$

5. Final remarks

We have shown that in contrast to the master equation, the filtering equation describing the reduction of the quantum state following the registered trajectory for a single and double heterodyne measurement does not destroy the squeezed coherent state (11). We have checked that for the initial state $|\xi_0, \alpha_0\rangle$ with $\xi_0 \neq 0$, the amount of squeezing decreases in time and registering the trajectory increases our knowledge about S . It is also worth to emphasize that for the system prepared initially in a coherent state, the prior and posterior mean values of the system operators coincide. The posterior coherent state has a random phase and decreasing in time independent of the noise amplitude. Even this case one can take advantage from using the filtering equation: the probability density $|l(t)|^2$ depends on α_0 , so $l(t)$ gives information on the initial state of the oscillator. The analytical solutions to the filtering equation for the initial coherent state obtained for the diffusion observation in [5, 16, 21], and for the counting process in [22] are consistent with our results. Finally, let us note that it will be very interesting to consider the posterior evolution of a squeezed coherent state when a driving force is applied to the oscillator.

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